

BARRIERS IN QUANTUM GRAVITY ¹

J. AMBJØRN

*The Niels Bohr Institute**Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark***ABSTRACT**

I discuss recent progress in our understanding of two barriers in quantum gravity: $c > 1$ in the case of 2d quantum gravity and $D > 2$ in the case of Euclidean Einstein-Hilbert gravity formulated in space-time dimensions $D > 2$.

1. $c > 1$

In the last four years there has been a tremendous progress in our understanding of 2d quantum gravity and non-critical strings. However, one of the main motivations for studying non-critical strings was to be able to formulate a string theory without tachyons for dimensions of target space larger than one, and in this respect there has been virtually no progress. Stated slightly differently: we have still not understood 2d quantum gravity coupled to matter with a central charge larger than one. If we use the discretized approach [1, 2, 4, 3, 5] there seems to be no formal problems with such a coupling. One has a statistical system of random surfaces coupled to a number of gaussian scalar fields or coupled to a number of spin systems, which only interact via the geometry of the random surface. Formally the system is reflection positive and there should be no tachyonic excitations in the system. However, two theorems established soon after the invention of these models made it unlikely that interesting theories would exist for $c > 1$. Recent results of numerical nature seem

¹Invited talk presented at the ITCP spring school and workshop on string theory, Trieste, Italy, April 1993. To appear in the proceedings.

to indicate loopholes related to the theorems [6, 7, 8, 9, 10, 11, 12]. In the following I will review the theorems and explain my present understanding of the loopholes.

1.1 *The theorems*

The following theorem was proven in the context of 2d gravity coupled to multiple Gaussian fields [2, 5, 13, 21], but the proof can presumably be generalized to other matter fields:

theorem 1 $\gamma \leq \frac{1}{2}$ and $\gamma > 0 \Rightarrow \gamma = \frac{1}{2}$.

theorem 2 $\sigma(\mu_c) > 0$.

In the first theorem γ denotes the string susceptibility. In the second theorem $\sigma(\mu)$ denotes the string tension as a function of the cosmological constant μ , and μ_c is the critical value of the cosmological constant, where the continuum limit is taken. Theorem 2 is only relevant for multiple Gaussian fields since these can be given the interpretation as target space coordinates of the string. The string tension is defined by the exponential decay of the one-loop function, the loop being a fixed planar boundary ∂A enclosing an area A . Let $G_\mu(\partial A)$ denote the one-loop Greens function. Then $\sigma(\mu)$ is defined by

$$G_\mu(\partial A) \sim e^{-\sigma(\mu)A}, \quad \sigma(\mu) \equiv \lim_{A \rightarrow \infty} \frac{\log G_\mu(\partial A)}{A}. \quad (1)$$

According to the KPZ formula γ is an increasing function of c , increasing to 0 as c increases to 1, while the formula leads to complex γ 's for $c > 1$. It is natural to interpret the first theorem as an indication that the random surfaces degenerate to so-called branched polymers as soon as $c > 1$, since the value $\gamma = 1/2$ is the generic value for branched polymer surfaces. This interpretation is further corroborated by theorem 2 in the case of multiple Gaussian fields provided we define the relation between the bare (dimensionless) string tension $\sigma(\mu)$ and the “continuum” string tension σ_{phys} having dimension (mass)² in the usual way. Let a be a scaling parameter with dimension of length, usually identified a typical lattice cut-off in the regularized theory. We imagine that a is a function of the cosmological constant which scales to 0 for $\mu \rightarrow \mu_c$, and defined with reference to some mass parameter of the theory which scales to zero. One choice could e.g. be the mass defined by the exponential decay of the two-point function in target space (i.e. the “tachyon” mass in ordinary string theory):

$$m(\mu) \sim const. \cdot (\mu - \mu_c)^\nu = m_{phys} a(\mu). \quad (2)$$

If it is not possible to find a bare mass which scales to zero, one would be tempted to conclude that the theory has no continuum interpretation in target space. If we *assume* the existence of a bare mass which scales according to (2) we get $a(\mu) \sim (\mu - \mu_c)^\nu$, and the physical mass m_{phys} is simply *our* choice of ratio between $m(\mu)$ and $a(\mu)$. The relation between the physical string tension σ_{phys} and the bare string $\sigma(\mu)$ tension has to be defined in a similar way:

$$\sigma(\mu) = \sigma_{phys} a^2(\mu). \quad (3)$$

Since $a(\mu)$ by assumption scales to zero we conclude that $\sigma_{phys} \rightarrow \infty$ for $\mu \rightarrow \mu_c$, due to theorem 2. The only excitations of a surface with very large string tension are those which do not increase minimal the area dictated by the boundary conditions. The only possible outgrowths will be thin tubes which are allowed to branch, i.e. precisely branched polymer surfaces.

1.2 Theorem 1

As mentioned above there has been accumulating evidence that especially theorem 1 might be circumvented in models with various kind of matter coupled to 2d quantum gravity. Let me mention possible loopholes in the proof of this theorem²

Let us for notational simplicity consider pure 2d gravity. In that case we know of course that $c = 0$ and $\gamma = -1/2$ but we can apply the general arguments to the model which is defined as follows:

$$Z(\mu) = \sum_{T \in \mathcal{T}} e^{-\mu|T|} \quad (4)$$

where \mathcal{T} denotes a suitable class of triangulations. We will here restrict the topology to be spherical. Let us further define the n -puncture function as³

$$G_n(\mu) = (-1)^n \frac{d^n}{d\mu^n} Z(\mu). \quad (5)$$

We have

$$G_n(\mu) \geq G_2^n(\mu), \quad n > 2. \quad (6)$$

The geometrical interpretation of eq. (6) is as follows: The rhs of eq. (6) can be viewed as a surface made out of n elongated balloons each with two marked points: one at the “root” and one at the top of the balloon, all tied together at their “roots”, this common point being viewed as an interior point. In this way

²This analysis was first performed by B. Durhuus in the case of multiple Ising spins, and most of the following discussion of theorem 1 is inspired by discussions with him.

³I order not to make the discussion too technical we have chosen not to specify certain symmetry factors in eq. (4) and eq. (5).

the rhs can be viewed as a surface with n marked points and this restrict sum over surfaces with n marked points is smaller than the sum over all surfaces with n marked points, i.e. the lhs of eq. (6). These arguments can be made rigorous in specific models (see for instance [13]).

By definition of the string susceptibility we have

$$G_n(\mu) \sim a.p. + (\mu - \mu_c)^{2-n-\gamma} + \dots \quad (7)$$

where $a.p.$ denotes an analytic part at μ_c while \dots denote less singular terms. From (6) we get

$$\gamma < 1 - \frac{1}{n-1} \quad (8)$$

and the strongest bound on γ comes from $n = 3$ and is $\gamma \leq 1/2$. It is tempting to use (8) for $n = 2$, in which case we would get $\gamma \leq 0$. However, (6) is not correct in that case due to a non-unique decomposition of surfaces in a sequence of “balloons”-like surfaces.

We can derive an equation in the case where $n = 2$ as follows: Let us consider the one-point function. To be more precise we will consider the function where one triangle is considered the external boundary. This function has the same scaling behaviour as the one-point function where only a vertex is kept fixed. Assume that the smallest allowed loops in our class of triangulations are of length 3, i.e. are triangles. Any 3-loop of links in the surface is either a triangle belonging to the surface or it can be viewed as the boundary of a surface which grows out from our the rest of the surface and which can be separated from the rest by cutting along the 3-loop. If we still use the notation “one-point” function for the surfaces with a triangle (a 3-loop) as boundary we can write:

$$G_1(\mu) = \sum_{T \in \tilde{\mathcal{T}}} e^{-\mu|T|} (1 + G_1(\mu))^{|T|} \equiv \sum_{T \in \tilde{\mathcal{T}}} e^{-\bar{\mu}|T|} \quad (9)$$

where $\tilde{\mathcal{T}}$ denotes the class of triangulations which cannot be separated in two by cutting along a loop of length 3. Clearly this class of triangulations is a perfectly good one by its own right and we can define a partition function and n -point functions similar to (4)-(5), only with $\tilde{\mathcal{T}}$ instead of \mathcal{T} . Let us denote the corresponding n -point functions with $\bar{G}_n(\bar{\mu})$, where $\bar{\mu}$ denotes the cosmological constant of the modified model. By definition we have

$$\bar{\mu} = \mu - \log(1 + G_1(\mu)), \quad \bar{G}_1(\bar{\mu}) = G_1(\mu). \quad (10)$$

We can differentiate these equations with respect to μ :

$$\frac{d\bar{\mu}}{d\mu} = 1 + \phi(\mu), \quad \phi(\mu) \equiv \frac{G_2(\mu)}{1 + G_1(\mu)} \quad (11)$$

and

$$\phi(\mu) = \frac{\bar{\phi}(\bar{\mu})}{1 - \bar{\phi}(\bar{\mu})}. \quad (12)$$

ϕ is essentially the two-point function G_2 . They have the same critical behaviour and we will not distinguish between them. $\bar{\phi}$ is defined as ϕ , just using \bar{G}_1 , \bar{G}_2 and $\bar{\mu}$ instead of G_1 , G_2 and μ . It can be viewed as the sum over surfaces which connect the two boundary points (or boundary 3-loops) and where the surfaces in the sum have the property that they cannot be separated in two components, each of which contains one of the boundaries, by a cut along a 3-loop. By expanding the denominator eq. (12) gets the obvious graphical interpretation of the two-point function $\phi(\mu)$ as the sum over all surfaces (of trivial topology) made up by successive gluing of “irreducible” $\bar{\phi}$ -surfaces along their boundaries, such that only two boundaries are left as external.

If we assume that $\gamma > 0$, we know from (7) that $\phi(\mu)$ will diverge for $\mu \rightarrow \mu_c$. From (12) this is impossible for $\bar{\phi}(\bar{\mu}(\mu))$ since it follows that

$$\bar{\phi}(\bar{\mu}_0) = 1, \quad \bar{\mu}_0 \equiv \bar{\mu}(\mu_c) \quad (\geq \bar{\mu}_c). \quad (13)$$

We conclude that either $\bar{\mu}_0 > \bar{\mu}_c$, the critical point for the modified model defined by the class of triangulations $\bar{\mathcal{T}}$, or $\bar{\gamma}$ corresponding to $\bar{\mathcal{T}}$ has to satisfy $\bar{\gamma} \leq 0$ in order that $\bar{\phi}(\bar{\mu}_c)$ is bounded (in fact $\bar{\phi}(\bar{\mu}_c) = 1$ by (12)).

In the first case we can expand $\bar{\phi}(\bar{\mu})$ in a Taylor series around $\bar{\mu}_0$ and use that due to (11) we have

$$\bar{\mu}_0 - \bar{\mu} \approx \left. \frac{d\bar{\mu}}{d\mu} \right|_{\mu} (\mu_c - \mu) \approx -(\mu - \mu_c)^{1-\gamma} \quad (14)$$

$$\bar{\phi}(\bar{\mu}) \approx 1 + \bar{\phi}'(\bar{\mu}_0)(\bar{\mu} - \bar{\mu}_0) \approx 1 + \bar{\phi}'(\bar{\mu}_0)(\mu - \mu_c)^{1-\gamma}. \quad (15)$$

If we insert $\phi(\mu) \approx (\mu - \mu_c)^{-\gamma}$ and (15) for $\bar{\phi}$ in eq. (12) we get the promised result $\gamma = 1/2$, i.e. the exponent corresponding to branched polymers.

In the Taylor expansion around $\bar{\mu}_0$ we used that $\bar{\phi}'(\bar{\mu}_0) < 0$. This follows from the fact that the two-point function is the sum over surfaces with positive weight $\exp(-\bar{\mu}|T|)$. If we however lift the restriction that the surfaces should enter with positive weight, by adding to the definition (4) a weight $\rho(T)$ which can take positive and negative values according to the triangulation, it is possible to arrange that

$$\bar{\phi}'(\bar{\mu}_0) = \dots = \bar{\phi}^{(n-1)}(\bar{\mu}_0) = 0, \quad \bar{\phi}^{(n)}(\bar{\mu}_0) \neq 0. \quad (16)$$

In this case we get instead of eq. (15)

$$\bar{\phi}(\bar{\mu}) \approx 1 + \bar{\phi}^{(n)}(\bar{\mu}_0)(\bar{\mu} - \bar{\mu}_0)^n \approx 1 + \bar{\phi}^{(n)}(\bar{\mu}_0)(\mu - \mu_c)^{(1-\gamma)n}. \quad (17)$$

If we insert this in eq. (12) we get

$$\gamma = 1 - \frac{1}{n+1}. \quad (18)$$

We can clearly arrange that $\gamma > 1/2$. But it is unlikely that the theory is different from a theory of branched polymers. In fact there exists a theory of so-called multicritical branched polymers, where the probability for branching can be negative [14]. The possible values of γ for such branched polymers are precisely given by (18).

The only possibility to get something non-trivial for $\gamma > 0$ seems to be the situation where $\bar{\mu}_0 = \bar{\mu}_c$. In this case we know that $\bar{\gamma} \neq \gamma$. In fact we have to have $\bar{\gamma} \leq 0$ since $\bar{\phi}(\bar{\mu}_c) = 1$. We can no longer make a Taylor expansion as in (15). We can however use (14) and the fact that $\bar{\phi}(\bar{\mu})$ (at least for $\bar{\gamma} \geq -1$) by the definition of $\bar{\gamma}$ behaves like

$$\bar{\phi}(\bar{\mu}) \approx 1 + (\bar{\mu} - \bar{\mu}_c)^{-\bar{\gamma}} \approx 1 + (\mu - \mu_c)^{-\bar{\gamma}(1-\gamma)}. \quad (19)$$

If we insert this in (12) we get

$$\gamma = -\frac{\bar{\gamma}}{1-\bar{\gamma}}, \quad \bar{\gamma} = -\frac{1}{n} \Rightarrow \gamma = \frac{1}{n+1} \quad (20)$$

which gives a remarkable relation between γ and $\bar{\gamma}$, first obtained by Durhuus [15]. The interpretation of this relation is however a little disappointing. We see that it might be possible to obtain a γ between 0 and 1/2. However, it does not really reflect a new theory. It seems rather to be a “polymer-like” or “bubble-like” manifestation of an underlying $c < 0$ theory.

The above analysis only opened for the possibility that $0 < \gamma < 1/2$. It did not provide any examples. The treatment was also simplified in the sense that we used the notation of pure gravity where we know by explicit calculation that $\gamma < 0$. There is an example of a model which realizes precisely the scenario mentioned above, a modified matrix model for generating random surfaces, first introduced in [16] and analysed further in [17, 18]. The partition function (in the sense of generating connected random surfaces) is given by

$$Z(\mu, g) = \log \left\{ \int dM \exp \left[-\mu N \text{Tr} V(M) + g\mu^2 (\text{Tr} M^2)^2 \right] \right\}. \quad (21)$$

where M denotes an $N \times N$ Hermitian matrix and $V(M)$ is a potential corresponding the n^{th} multicritical matrix model. If $g = 0$ this matrix model will generate random surfaces with $\gamma = -1/n$. We have a critical line in the (μ, g) -coupling constant plane and at a certain point along this line the critical behaviour changes

and γ jumps from $-1/n$ to $1/(n+1)$. The geometrical interpretation of the model is that the term $(\text{Tr } M^2)^2$ introduces a new kind of vertex in the model, such that to leading order in $1/N$ ordinary spherical surfaces dominates, except that two (or more) such surfaces can touch each other by means of the new vertices, the topology of the total surface still being spherical. The jump in γ is related to a change in behaviour the effective “touching” coupling constant $N\bar{g} = -g\mu < \text{Tr } M^2 >$ as a function of μ , from being analytic to non-analytic, i.e. a situation precisely as described above in general terms.

The other class of models where one could imagine that $0 < \gamma < 1/2$ is multiple Ising models coupled to gravity. Durhuus has analysed this situation along the lines indicated above[15]. Things are slightly more complicated since we will now have several coupling constants and the scalar equations written above will be replaced by matrix equations. However, the result is unchanged: The only possible alternative to $\gamma = 1/2$ seems to be that $\gamma = 1/(n+1)$, and this model is somehow a shadow of an underlying unitary model with $\bar{\gamma} = -1/n$ and a central charge $\bar{c} < 1$.

There is some numerical evidence in favor of such a scenario [8, 9], but since it is now more clear what to look after it should be possible to provide much better numerical verification of the above scenario for models with $c > 1$.

1.3 Theorem 2

Theorem 2 is more specific for discretized string models, and the loopholes not obvious, if any. One cure, which is inspired by the geometrical picture of spiky polymers emerging from a minimal surface, attempts to add by hand extrinsic curvature terms. The model was first formulated in [19, 20], partly inspired by theorem 2 ([21]), partly by the fact that one could get extrinsic curvature terms by integrating out the fermionic part of the superstring. It seems difficult to obtain analytic results for the model (see however [22] for a renormalization group approach), and most work has been of numerical nature. From large scale Monte Carlo simulations the following have been established (with the usual reservation of numerical simulations, which have a lot in common with experimental physics, including the ability to produce results which have to be retracted eventually): As a function of the extrinsic curvature coupling constant λ we have two phases : One phase where the physics is the same as for $\lambda = 0$, i.e. the string tension $\sigma(\mu, \lambda)$ is not scaling to zero as $\mu \rightarrow \mu_c(\lambda)$ (the critical value of the cosmological constant μ now depends on λ), and , for $\lambda > \lambda_c$, a now phase where the surfaces essentially are “smooth” embeddings in target space with Hausdorff dimension $d_H = 2$. *At the transition point* λ_c we have according to the numerical simulations [23, 24, 25] that the string tension scales to zero as $\mu \rightarrow \mu_c(\lambda_c)$. In addition the mass gap $m(\mu, \lambda_c)$

of the two-point function scales to zero for $\mu \rightarrow \mu_c(\lambda_c)$ in such a way that

$$\frac{m^2(\mu, \lambda_c)}{\sigma(\mu, \lambda_c)} \approx \text{constant}. \quad (22)$$

According to the discussion above this is precisely what is needed in order to achieve an interesting continuum limit which allow the interpretation as a string theory. It is, however, at present unclear which continuum theory could serve as a candidate for the continuum limit of the regularized string theory.

It is interesting to view the model with extrinsic curvature in the framework of conformal field theory coupled to 2d quantum gravity. From this point of view we should first consider a fixed regular triangulation. The lattice points are mapped to target space (say R^3 , where most of the numerical simulations are actually performed) and the action is the usual Gaussian action plus extrinsic curvature (for details we refer i.e. to [26, 27]). It is local, but non-polynomial in the target space coordinates x_i , due to the extrinsic curvature term. The model has also attracted the attention of the solid state physicists and it is known that it has an infrared fixed point at $\lambda = 0$ and a ultraviolet fixed point at a certain critical value of λ . The central charge for $\lambda = 0$ is 3, since we in that case have three free gaussian fields. The central charge corresponding to the ultraviolet fixed point is unknown. In case of a unitary theory one would (by the c -theorem) expect $c \geq 3$, but it is not known if the conformal field theory corresponding to the ultraviolet fixed point is unitary. Coupling to gravity should now, according to the standard procedure, consist in replacing the regular triangulation with dynamical triangulations. The phase transition reported above for dynamical triangulations is precisely the one analogue of the transition corresponding to the UV-fixed point for the regular triangulation. Have we manage to couple a theory with $c > 1$ to quantum gravity with a non-trivial result? Certainly this scenario deserves further attention.

2. QUANTUM GRAVITY FOR $D > 2$

Attempts to quantize quantum gravity in dimensions $D > 2$ differ radically from the attempts described above for $D = 2$. For $D > 2$ we have not a well understood theory when the rotation to Euclidean space is performed: Due to the conformal mode the Einstein-Hilbert action is unbounded from below. The Euclidean path integral is not well defined. One way to avoid this problem was suggested by Hawking: An analytical continuation of the conformal mode. But it is not known if the method will work beyond perturbation theory (see [29] for an example where analytic continuation works to all order in perturbation theory but fails non-perturbatively). Another suggestion is to use stochastic quantization [28]. Also this method has the flavor of being somewhat arbitrary. In addition to the problems with the conformal

mode we are in quantum gravity faced with the question of topology. Should all topologies be included in the quantum theory, and if this is the case have can we imagine this should be done, since it is impossible to classify 4-geometries?

Contrary to two dimensions we are obviously in the situation where we try to define in a non-perturbative way a theory, which existence can be said to be somewhat doubtful. The only reason we engage in such a project is the fact that we know from experimental evidence that there exists *a theory of gravity*, which presumably should be quantized. One candidate for a non-perturbative definition is of course string theory, but since it is presently by no means obvious that string theory itself exists in a non-perturbative formulation, one can seriously doubt that it can be used define quantum gravity in a non-perturbative way.

Simplicial quantum gravity is an attempt to give a non-perturbative definition of Euclidean gravity. Its main advantage is that it has been shown to work well (at least as well as any method based on a continuum formulation) in $D = 2$. In addition it provides a very natural geometric discretization of the Einstein-Hilbert action. A drawback is that it deals in a superficial way with the conformal mode problem⁴ since the regularization is such that for a fixed lattice volume the action is bounded from below. Of course this just postpone the conformal mode problem until the scaling limit has to be taken. In addition it has little to say about the summation over topologies, but at least it highlights the problem, since the model is simply not well defined unless we restrict the topology. In the following we will always assume the simplest topology, i.e. S^4 in $D = 4$.

2.1 The model

As in $D = 2$ we imagine the manifolds which enters in the regularized path integral to be constructed by gluing together identical simplexes, in this case 4-simplexes rather than the equilateral triangles used in $D = 2$. The curvature of such a piecewise linear manifold is given by Regge calculus, which in this case becomes extremely simple (see for instance [30, 31] and reference herein): Suppose that our manifold is constructed out of N_4 simplexes and that it has N_2 two-dimensional sub-simplexes. Using the construction of Regge a little algebra leads to the following identifications (which we write down in D dimensions for the sake of generality):

$$\int d^D \xi \sqrt{g} \propto N_D \tag{23}$$

$$\int d^D \xi \sqrt{g} R \propto \frac{2\pi}{\arccos(1/D)} N_{D-2} - \frac{D(D+1)}{2} N_D. \tag{24}$$

⁴This is however true for almost all regularizations.

For a given four-dimensional triangulation T we can therefore write the cosmological term plus the Einstein-Hilbert action as

$$S[T] = \kappa_4 N_4(T) - \kappa_2 N_2(T) \quad (25)$$

where κ_2 is proportional to the inverse of the bare gravitational coupling constant and κ_4 involves a combination of the bare cosmological coupling constant and the bare gravitational coupling constant.

The formal recipe for going from the continuum functional integral to the discretized one is now:

$$\int \mathcal{D}[g_{\mu\nu}] \rightarrow \sum_{T \in \mathcal{T}} \quad (26)$$

$$\int \mathcal{D}[g_{\mu\nu}] e^{-S[g]} \rightarrow \sum_{T \in \mathcal{T}} e^{-S[T]} \quad (27)$$

where $[g_{\mu\nu}]$ denotes equivalence classes of metrics, i.e. the volume of the diffeomorphism group is divided out, and it is understood that the functional integral is only over connected four-manifolds of topology S^4 .

It is straight forward to generalize the action to include the coupling to matter, e.g. spin systems or Gaussian fields (we refer to [32] for details).

The rhs of eq. (27) shares a number of properties with its two-dimensional analogue: The number of triangulations of topology S^4 and a given volume N_4 is exponentially bounded⁵. This implies that the partition function for a given κ_2 has a critical κ_4 where the infinite volume limit can be taken. It does not imply that we necessarily have an interesting continuum theory. One has to scan κ_2 to look for divergent correlation lengths, at least if we want to use the standard intuition from statistical mechanics and the theory of critical phenomena. Unfortunately the only tool which seems useful at the moment in the analysis of the scaling limit is numerical simulations.

2.2 Observables and numerical simulations

Due to the invariance under diffeomorphism and the fact that we in quantum gravity have to integrate over all Riemannian manifolds, the observables which are most readily available are averages of invariant local operators like the curvature $R(x)$. A non-local observable is the integrated curvature-curvature correlation. In a continuum formulation it would be

$$\chi(\kappa_2) \equiv \langle \int d^4\xi_1 d^4\xi_2 \sqrt{g(\xi_1)} R(\xi_1) \sqrt{g(\xi_2)} R(\xi_2) \rangle - \langle \int d^4\xi \sqrt{g(\xi)} R(\xi) \rangle^2. \quad (28)$$

⁵This is only verified by numerical methods. It would be interesting to have an analytic proof of this.

In a lattice regularized theory one would expect that away from the critical points short range fluctuations will prevail, while approaching the critical point long range fluctuation might be important and would result in an increase of $\chi(\kappa_2)$. The observable $\chi(\kappa_2)$ is the second derivative of the free energy $F = -\ln Z$ with respect to the inverse gravitational coupling constant. In the case where the volume N_4 is kept fixed we have

$$\chi(\kappa_2, N_4) \sim \langle N_2^2 \rangle_{N_4} - \langle N_2 \rangle_{N_4}^2 = - \frac{d^2 \ln Z(\kappa_2, N_4)}{d\kappa_2^2} \quad (29)$$

From the above discussion it follows that we have to search for values of κ_2 where $\chi(\kappa_2, N_4)/N_4$ diverges in the infinite volume limit $N_4 \rightarrow \infty$.

An independent susceptibility is associated with volume fluctuations:

$$\chi_V(\kappa_2, \kappa_4) \sim \langle N_4^2 \rangle - \langle N_4 \rangle^2 = - \frac{d^2 \ln Z(\kappa_2, \kappa_4)}{d\kappa_4^2} \quad (30)$$

Assume that $Z(\kappa_2, N_4)$ has the form

$$Z(\kappa_2, N_4) \sim N_4^{\gamma(\kappa_2)-3} e^{\kappa_4^c(\kappa_2)N_4} (1 + O(1/N_4)). \quad (31)$$

This is the case for $D = 2$. For higher D it is of course a necessity for the existence of the model that $Z(\kappa_2, N_4)$ is exponentially bounded, but it is by no means clear that subleading corrections should appear as a power-law correction to the exponential factor. If it nevertheless does, we can identify $\gamma(\kappa_2)$ with the critical exponent for the volume fluctuations at the critical point $\kappa_4^c(\kappa_2)$:

$$\chi_V(\kappa_2, \kappa_4) \sim \frac{1}{(\kappa_4 - \kappa_4^c(\kappa_2))^{\gamma(\kappa_2)}} \quad (32)$$

for κ_4 close to κ_4^c .

Another observable is the Hausdorff dimension. One can define the Hausdorff dimension in a number of ways, which are not necessarily equivalent. Here we will simply measure the average volume $V(r)$ contained within a radius r from a given point. In [33] the concept of a *cosmological Hausdorff dimension* d_{ch} was defined. It essentially denotes the power which relates the average radius of the ensemble of universes of a fixed volume to this volume:

$$\langle \text{Radius} \rangle_{N_4} \sim N_4^{1/d_{ch}}. \quad (33)$$

From the distribution $V(r)$ we can try to extract d_{ch} . If for large r we have the behaviour

$$V(r) \sim r^{d_h} \quad (34)$$

we can identify d_{ch} and d_h .

By the use of Regge calculus it is straightforward to convert these continuum formulas to our piecewise flat manifolds. Let me now summarize some of the numerical results obtained with the last $1\frac{1}{2}$ year by various groups [34, 35, 36, 30, 38].

The measurements of $\chi(\kappa_2)$ show a peak at $\kappa_2 \approx 1.1$. We will denote this value κ_2^c . At this point there indeed seems to be some kind of phase transition in geometry. For $\kappa_2 < \kappa_2^c$ the radius of the typical universes generated by computer is very small, and the radius grows very slowly with the volume N_4 , indicating a large fractal dimension of space-time. For $\kappa_2 > \kappa_2^c$ the situation is the opposite: The radius of the typical universe is large, in fact much larger than one would naively expect, and there is a strong dependence on the volume N_4 . A closer analysis of the distribution of distances reveals that the geometrical structures generated are of low fractal dimension (between 1 and 2). If we approach κ_2^c it seems that we have some chance of generate structures with a fractal dimension which is not that different from 4, but it is difficult to measure the Hausdorff dimension precisely. At a first glance κ_2^c seems to be the obvious candidate for a critical point where we can take the continuum limit. There is one obstacle, however. It turns out that the scalar curvature does not scale at the transition point. Since the continuum curvature has the dimensions of $(\text{mass})^2$, standard scaling arguments will tell us that that the bare curvature has to scale to zero at the critical point. This seems not to be the case in the numerical simulations⁶ and the manifolds which are generated can therefore not in any simple way be identified with smooth continuum manifolds!

In order to make contact to various continuum models for quantum gravity it is important to measure critical exponents which can be calculated in a continuum approach. One such exponent is $\gamma(\kappa_2)$, the exponent of volume fluctuations, defined by eq. (32). A convenient way of performing the measurement of $\gamma(\kappa_2)$ is by counting so-called baby universes [37, 8]. These are just parts of the triangulated piecewise linear manifold which can be separated from the rest by cutting the manifold along a closed, minimal three-dimensional submanifold, i.e. the hyper-surface of a 4-simplex. In most cases such a three-manifold *will* just be the hyper-surface of a 4-simplex which belong to the triangulation. But there exists situations where the “interior” of the hyper-surface does not belong to the triangulation⁷ and in this case the universe will be separated in two smaller universes of volume B and $N_4 - B$, where N_4 is the original volume of the universe and we assume $B < N_4/2$.

⁶This problem may be similar in nature to the non-scaling of the string tension mentioned above.

⁷The two-dimensional analogue is to cut a surface along a 3-loop. In most cases one will just cut off a triangle, but if the three loop is a “bottleneck” separating two “blobs” of surfaces we have a genuine baby-universe situation.

We call B the volume of the baby universe and it can be shown that under the assumption of a distribution like (31), the distribution of baby universes will be

$$n(B) \sim N_4 B^{\gamma(\kappa_2)-2}, \quad B \ll N_4. \quad (35)$$

The results of measuring the distribution of baby universes indicate that the assumption of a distribution like (31) fails⁸ for $\kappa_2 < \kappa_2^c$ but is fulfilled for $\kappa_2 > \kappa_2^c$. This is in agreement with similar observations made for simplicial quantum gravity in $D = 3$ [39, 40, 41, 42]. The value extracted close to κ_2^c is approximately -0.5 ± 0.2 , i.e. close to the value for two-dimensional gravity!

There are various continuum candidate theories with which we can hope to compare the numerical simulations. One candidate is the restricted theory of self-dual conformal gravity. This theory allows an analysis by 2d conformal techniques [43] and an ultraviolet fixed point is found around which γ can be calculated ($\gamma = -0.67$) and it is found to agree within errorbars with the value measured at κ_2^c . Another model which is somewhat similar in spirit to the self-dual conformal gravity in the sense that the physics of the conformal mode is emphasized is a mini-super space model described in [44]. It does not yet predict a value of γ , but the structure of the phase diagram has some similarity to the one found in simplicial quantum gravity. Finally a model of quantum gravity based on the $1/\epsilon$ expansion around $D = 2$ might be of interest as a regularized model which allow us to compute critical exponents[45, 46] and a comparison with the $\gamma(\kappa_2)$'s determined by numerical simulations will be an important task for the future.

Acknowledgement It is a pleasure to thank Bergfinnur Durhuus for explaining to me his ideas about $c > 1$ Ising spin models. They formed the foundation for section 1.2.

References

- [1] F. David, Nucl. Phys. **B257** (1985) 45.
- [2] J. Ambjørn, B. Durhuus and J. Fröhlich, Nucl. Phys. **B257** (1985) 433.
- [3] F. David, Nucl. Phys. **B257** (1985) 543.
- [4] V. A. Kazakov, I. K. Kostov and A. . Migdal, Phys. Lett. **157B** (1985) 295.
- [5] J. Ambjørn, B. Durhuus, J. Fröhlich and P. Orland, Nucl.Phys. **B270** (1986) 457.

⁸In [37] we extract γ 's for $\kappa_2 < \kappa_2^c$, but it appears that the baby universe volumes used were too small to do it in a reliable way.

- [6] E. Brezin and S. Hikami, Phys.Lett. **B295** (1992) 209.
- [7] S. Hikami, Phys.Lett. **B305** (1993) 327.
- [8] J. Ambjørn, S. Jain and G. Thorleifsson, Phys.Lett. **B307** (1993) 34.
- [9] J. Ambjørn, B. Durhuus, T. Jonsson and G. Thorleifsson, Nucl.Phys. **B398** (1993) 568.
- [10] M. Wexler, *Matrix models on large graphs.*, PUPT-1398, Bulletin Board: hep-th@xxx.lanl.gov - 9305041; *Low temperature expansion of matrix models.*, PUPT-1384, Bulletin Board: hep-th@xxx.lanl.gov - 9303146
- [11] C.A. baillie and J.D. Johnston, Phys.Lett. **B286** (1992) 44.
- [12] S. M Catterall, J.B. Kogut and R.L. Renken, Phys.Lett. **B292** (1992) 277.
- [13] J. Ambjørn, B. Durhuus and J. Fröhlich, Nucl.Phys. **B275** [FS17] (1986) 161.
- [14] J. Ambjørn, B. Durhuus, T. Jonsson, Phys.Lett. **B244** (1990) 403
- [15] B. Durhuus, unpublished.
- [16] y S.R. Das, A. Dhar, A.M. Sengupta and S.R. Wadia, Mod.Phys.Lett. **A5** (1990) 1041.
- [17] y G.P. Korchemsky Phys.Lett. **B296** (1992) 323; *Matrix models perturbed by higher curvature terms*, UPRF-92-334.
- [18] y L. Alvarez-Gaume, J.L.F. Barbon, and C. Crnkovic, Nucl.Phys. **B394** (1993) 383.
- [19] J. Ambjørn, B. Durhuus, T. Jonsson, Phys.Rev.Lett. **58** (1987) 2619.
- [20] J. Ambjørn, B. Durhuus, J. Fröhlich and T. Jonsson, Nucl.Phys. **B290** [FS20] (1987) 480.
- [21] J. Ambjørn and B. Durhuus, Phys.Lett **188B** (1987) 253.
- [22] J. Ambjørn, B. Durhuus, J. Fröhlich and T. Jonsson, J. Stat. Phys. **55** (1989) 29.
- [23] S.M. Catterall, Phys.Lett **B243** (1990) 121.
- [24] J. Ambjørn, A. Irbäck, J. Jurkiewicz and B. Petersson, Nucl. Phys. **B393** (1993) 571.
- [25] J. Ambjørn, A. Irbäck, J. Jurkiewicz, B. Petersson and S. Varsted, Phys.Lett **B275** (1992) 295.
- [26] J. Ambjørn, B. Durhuus, T. Jonsson, Nucl. Phys. **B316** (1989) 526.
- [27] R. G. Harnish and J.F. Wheeler, Nucl.Phys. **B350** (1991) 861.

- [28] J. Greensite and M. Halpern, Nucl.Phys. **B242** (1984) 167.
- [29] J. Ambjørn and J. Greensite, Phys.Lett. **B254** (1991) 66.
- [30] J. Ambjørn, J. Jurkiewicz and C.F. Kristjansen, Nucl.Phys. **B393** (1993) 601.
- [31] J. Ambjørn, Z. Burda, J. Jurkiewicz and C.F. Kristjansen, Acta Physica Polonica **B23** (1992) 991.
- [32] J. Ambjørn, Z. Burda, J. Jurkiewicz and C.F. Kristjansen, *4d quantum gravity coupled to matter* NBI-HE-93-3, January 1993.
- [33] J. Ambjørn, B. Durhuus, T. Jonsson, Mod.Phys.Lett. **A6** (1991) 1133.
- [34] J. Ambjørn and J. Jurkiewicz, Phys.Lett **B278** (1992) 42.
- [35] M.E. Agishtein and A.A. Migdal, Nucl.Phys. **B385** (1992) 395.
- [36] M.E. Agishtein and A.A. Migdal Mod.Phys.Lett. **A7** (1992) 1039.
- [37] J. Ambjørn, S. Jain, J. Jurkiewicz and C.F. Kristjansen, Phys.Lett. **B305** (1993) 208.
- [38] B. Brüggmann, *Non-uniform Measure in Four-Dimensional simplicial Quantum Gravity*, SU-GP-92/9-1; B. Brüggmann and E. Marinari, *4d Simplicial Quantum Gravity with a Non-Trivial Measure*, SU-GP-92/9-2.
- [39] J. Ambjørn and S. Varsted, Phys. Lett. **B226**, (1991) 285.
- [40] D. Boulatov and A. Krzywicki, Mod.Phys.Lett. **A6** (1991) 3005.
- [41] J. Ambjørn and S. Varsted, Nucl.Phys. **B373** (1992) 557.
- [42] J. Ambjørn, D. Boulatov, A. Krzywicki and S. Varsted, Phys.Lett **B276** (1992) 432.
- [43] C. Schmidhuber Nucl.Phys. **B390** (1993) 188.
- [44] I. Antoniadis, P.O. Mazur and E. Mottola, *Scaling behavior of quantum four - geometries*. CPTH-A214-1292. Bulletin Board: hep-th@xxx.lanl.gov - 9301002
- [45] H. Kawai and M. Ninomiya, Nucl.Phys. **B336** (1990) 115.
- [46] H. Kawai, Y. Kitazawa and M. Ninomiya Nucl.Phys. **B393** (1993) 280; *Ultraviolet stable fixed point and scaling relations in (2+epsilon)-dimensional quantum gravity*, UT-636. Bulletin Board: hep-th@xxx.lanl.gov - 9303123